An Overview of Linear Systems

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- Description

  This course provides an introduction to linear systems primarily with a view towards modeling, simulation, filtering, and control system design. The material introduces linear, time-invariant systems that can be modeled with ordinary, constant coefficient, differential equations. The Laplace transforms and transfer functions are used to simply the analysis. Bode and Nyquist plots are used to present the system frequency response.

- Module List:
  1) Modeling of continuous time invariant linear systems
  2) Analysis of linear systems, state space representation, numerical simulation
  3) Analysis of linear systems, control system design and synthesis
  4) Implementation of control systems, discretization, z-transforms.

- Author: Duane Mattern

- Background:

Duane Mattern is an independent contractor specializing in modeling, simulation, control system design and implementation. He is experienced with rapid prototyping software tools like the Mathwork's Matlab/Simulink/Controls/RTW and Integrated System's Xmath/MatrixX/SystemBuild/Autocode. As a mechanical engineer specializing in instrumentation and controls with more than 10 years of experience, he has a broad range of practical knowledge, including automatic testing machines, turbofan engine control, integrated flight and propulsion control, servo-systems including voice coil and electromagnetic actuation, diagnostics, and neural networks. His current interests are in embedded system programming for control system implementation, modeling, and real-time simulation.

- Prerequisites:

  Familiarity with the following concepts: (i) phasor notation and the fundamentals of complex variables; (ii) integration and differentiation; (iii) superposition, the Laplace transform and transfer functions; (iv) Frequency response using Bode and Nyquist plots.

- Intended Audience:

  (1) Engineer or practitioner who would like to renew their knowledge of linear systems;
  (2) Engineer or practitioners who would like a fast introduction to linear systems;
  (3) College student who desires an alternative presentation to linear systems, separate from what they receive in their normal courses.

Estimated Total Course Learning Time:

1-2 hours

Module 1: Basic Building Blocks

- Purpose:
  - To introduce linear systems modeling.
  - To describe the purpose of these models as engineering tools.
  - To introduce the basic modeling elements.

- Objective:
  - Identify the basic building blocks of linear time invariant systems.
  - Identify the underlying assumptions with these models.
  - Identify the analogies that exist between electrical, mechanical, fluid, and thermal systems.

- Contents: 7 pages
  2 test questions

- Learning Time: 20 minutes

This module explains the basic components used for constructing linear system models. Upon completion, you will be able to accomplish the objective listed here. Click the Forward arrow when you’re ready to continue.

Content Slide 1: Mathematical modeling and linear systems

Why Model?
- To gain understanding and to perform tests without expensive hardware.
- To design better products in less time using the knowledge gained from the model.

What are Linear System Models
- An approach to modeling that fits into a specific class of mathematical methods.
- We use constant coefficient, linear, ordinary differential equations.
- We transform the ordinary differential equations into algebraic equations using the Laplace Transform.

The purpose of mathematical modeling is to approximate the characteristics of a system. We can use these models to perform experiments on the system before going to hardware testing. This software analysis allows us to inexpensively gain a deeper understanding of the physics behind the system behavior. The end result is better a product that is designed in less time.

Linear systems models are an approach to approximating the characteristics of systems using constant coefficient, linear, ordinary differential equations. Rather than using the differential equations directly, we convert the ordinary differential equations into algebraic equations using the Laplace transform. This transforms the problem from the time domain to the frequency domain and the algebraic equations can be rearranged to show the output over input ratio properties of the system.

Content Slide 2: Classification of models

Differential equations

- Partial Differential equations: more than one independent variable, typically time and spatial dimensions. For example, consider heat conduction across a flat plate. Computational techniques like finite elements and computational fluid dynamics are used to address these types of problems.

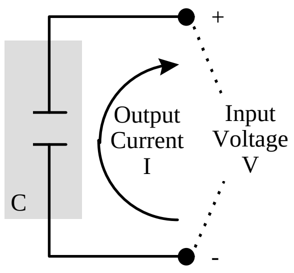
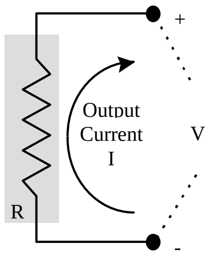
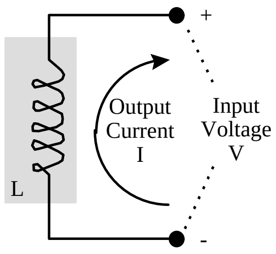
- Ordinary Differential equations: 1 independent variable, typically time and the spatial dimension is typically lumped.
  - Linear Ordinary Differential Equations. Superposition holds.
    - Constant Coefficient, Linear Ordinary Differential Equations
      - The initial value problem can be transformed into an algebraic problem using the Laplace transform.

(The Laplace transforms and transfer function sections)

This screen show where *Linear Systems* fit among a set of mathematical equations used for to model certain classes of systems. For example, finite element and computational fluid dynamics techniques have been developed to solve problems that are best described by partial differential equations. *Linear Systems* deals with the class of problems that are best described by linear ordinary differential equations. The scope of this online course will be limited to the problems that can be represented by constant coefficient linear ordinary differential equations. The point to be made is that Linear Systems is only a subset of a much larger domain.

To simplify our numerical problem, we will use the known initial conditions and the Laplace transform to convert the differential equation into an algebraic equation. The algebraic equation will be rearranged into an output over input transfer functions of the system. This transformation will allows us to use the tools associated with sinusoidal transfer functions in the frequency domain.

Content slide 3: Building blocks: capacitance, resistance, inductance.

Capacitance	Resistance	Inductance
		
(1) $I(t) = C\frac{dV(t)}{dt}$ **(2)** $\frac{I(s)}{V(s)} = G_c(s) = Cs$	**(3)** $\frac{I}{V} = \frac{1}{R}$ **(4)** $\frac{I(s)}{V(s)} = G_R(s) = \frac{1}{R}$	**(5)** $I(t) = \frac{1}{L}\int V(t)dt + I(0)$ **(6)** $\frac{I(s)}{V(s)} = G_L(s) = \frac{1}{Ls}$

The building blocks of linear system models of electrical systems are the resistor, capacitor and the inductor. Each of these idealized, lumped parameter elements, has an ordinary differential equation that describes the time domain behavior of the component. Assuming zero initial conditions and using the Laplace transform we can convert these ordinary differential equations to algebraic equations parameterized by the Laplace variable, "s". By defining an input and output, we can write a transfer function for the component as the output over the input ratio. Here we choose to define the voltage as the input and the current as the output.

The model of the ideal capacitor provide a time derivative relationship. Using the Laplace transform we can transform the differential equation (1) into an algebraic equation (2), where I(s) is the Laplace transform of the current, V(s) is the Laplace transform of the voltage and "s" is indicative of a time derivative.

The model of the ideal resistor does not depend on time and V=IR. The output over input transfer function in independent of s in equation (4).

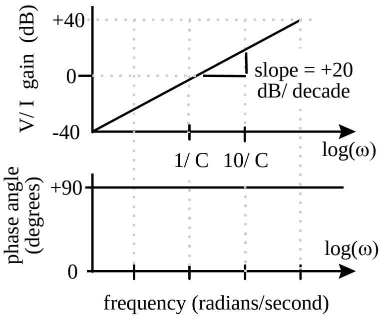
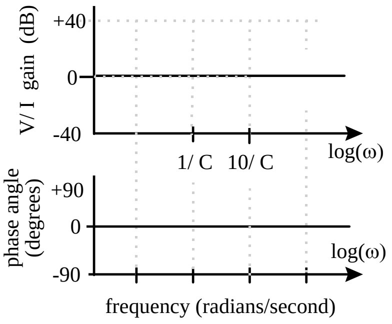
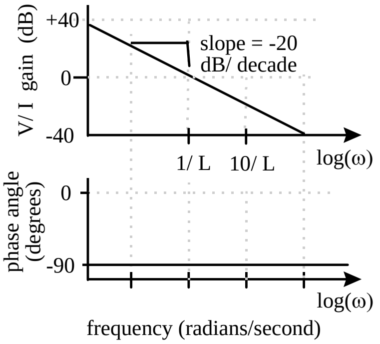
The model of the ideal inductor show an integral relation. Using the Laplace transform and assuming I(0)=0, this integral equation (5) is transformed into the algebraic equation (6), where "1/s" is indicative of integration.

Content slide 4: The Laplace Transform	
f(s) is the Laplace transform of *F(t)* $f(s) = \int_0^\infty e^{-st} F(t)dt = \mathcal{L}\{F(t)\}$	F(t)=1,for t > 0, then $f(s) = \int_0^\infty e^{-st} dt = \mathcal{L}\{1\} = \frac{1}{s}$
First derivative with respect to time	$\mathcal{L}\left\{F^1(t)\right\} = \mathcal{L}\left\{\frac{dF(t)}{dt}\right\} = s\mathcal{L}\{F(t)\} + F(0)$
“n”th order derivatives with respect to time.	$\mathcal{L}\left\{F^n(t)\right\} = \mathcal{L}\left\{\frac{d^n F(t)}{dt^n}\right\} = s\mathcal{L}\{F^{(n-1)}(t)\} + F^{(n-1)}(0)$
Linear Property	$\mathcal{L}\{aF(t) + bG(t)\} = \mathcal{L}\{aF(t)\} + \mathcal{L}\{bG(t)\}$
Integral with respect to time	$\mathcal{L}\left\{\int F(t)dt\right\} = \frac{f(s)}{s} + \frac{\int F(0)dt}{s}$

The Laplace transform operation, represented by a script capital L, “$\mathcal{L}$”, is an integral operation. The Laplace transform of a function exists if the Laplace integral converges. The Laplace transform of a function may be obtained through direct integration, but typically a table of Laplace transforms is used to find the result. If the desired function is not listed in the table, the properties associated with the Laplace transform usually can be use to construct a result out of the functions listed in the table.

In the following, we will be interested in the Laplace transform of derivatives and integrals of variables and the linearity properties of the Laplace transform. Also we will be assuming that the initial conditions are zero as part of the definition of transfer functions. Then, the Laplace transform of the derivative of a variable is just the Laplace transform of the variable times “s”. Similarly, the Laplace transform of the integral of a variable is just the Laplace transform of the variable divided by “s”.

The above description of the Laplace transform is not mathematically precise. Detailed description of the Laplace transform can be obtained from in textbooks on ordinary differential equations.

<table>
<tr><td colspan="3">Content slide 5: Bode plots of building blocks..</td></tr>
<tr><td>Capacitance
$$\frac{I(s)}{V(s)} = G_c(s) = Cs$$</td><td>Resistance
$$\frac{I(s)}{V(s)} = G_R(s) = \frac{1}{R}$$</td><td>Inductance
$$\frac{I(s)}{V(s)} = G_L(s) = \frac{1}{Ls}$$</td></tr>
<tr><td>
Bode Frequency Response</td><td>
Bode Frequency Response</td><td>
Bode Frequency Response</td></tr>
</table>

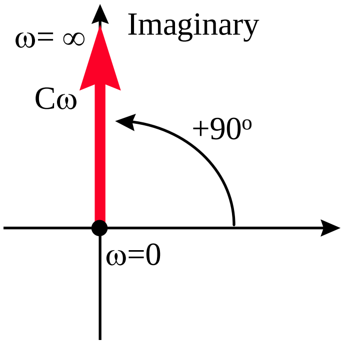
The Laplace variable “s”, is a complex variable, s=σ+jω, that is a function of frequency. For our application we will be interested in the jω axis, where ω is frequency in radians/sec . We can substitute jω for s in the sinusoidal transfer function and using complex arithmetic we can obtain a complex function of frequency, (ω) for the output over input ratio. There are a many ways to display this function. If we use phasor notation, we can convert the complex function into an amplitude ratio or gain and phase angle. The amplitude ratio at a specific frequency is the gain from a sinusoidal input at that frequency, to the resulting sinusoidal output at that same frequency. The phase angle at a specific frequency is the phase angle between the sinusoidal input at the given frequency to the resulting sinusoidal output at that same frequency. Linearity and superposition require the output signal to be at the same frequency as the input signal. This is a requirement of linear systems.

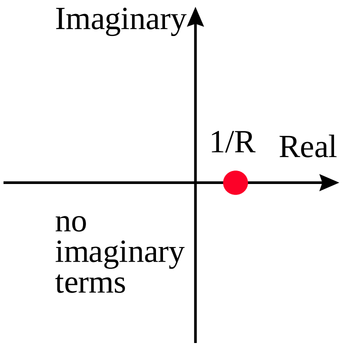
If we plot the amplitude and phase as a function of frequency, we obtain a Bode plot. The figures show the Bode plots for the models of the capacitor, resistor, and inductor. Note that the abscissa (x-axis) is plotted on a log scale. The amplitude is plotted in decibels (db) calculated as $20\log_{10}(\text{gain})$, {20 times the log base 10 of the gain}. You can see that the derivative in the capacitor model results in a slope of +20 db/decade. The integrator in the inductor results in a slope of -20 db/decade, where a decade is an order of magnitude increase in frequency, (ω to 10ω). The resistor amplitude ratio is “flat” at 0 db. Also note that that the current in the capacitor leads the voltage by 90 degrees and the current in the inductor lags the voltage by 90 degree.

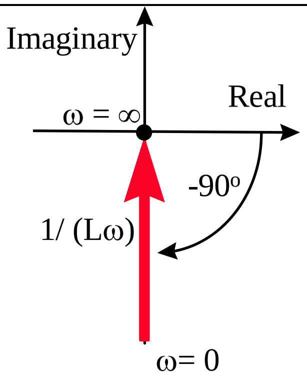
<table>
<tr><td colspan="3">Content slide 6: Nyquist plots of building blocks.</td></tr>
<tr><td>Capacitance

$$\frac{I(s)}{V(s)} = G_c(s) = Cs$$</td><td>Resistance

$$\frac{I}{V} = G_R(s) = \frac{1}{R}$$</td><td>Inductance

$$\frac{I(s)}{V(s)} = G_L(s) = \frac{1}{Ls}$$</td></tr>
<tr><td>

Nyquist Frequency Response</td><td>

Nyquist Frequency Response</td><td>

Nyquist Frequency Response</td></tr>
</table>

Alternatively, we can take the complex function of frequency that the transfer function represents and plot it in the complex plane with the x-axis as the real axis and the y-axis as the imaginary axis. This results in a Nyquist or direct polar plot. . The figures show the Nyquist plots for the idealized models of the capacitor, resistor, and inductor. Note that the traces in these plots are function of frequency and the positive phase angle is counter-clockwise.

The magnitude or gain for the ideal capacitor model from input voltage to output current is zero at zero frequency and large at large frequencies.

The magnitude of the ideal resistor model is constant for all frequencies.

The magnitude or gain for the ideal inductor transfer function from input voltage to output current is infinite at zero frequency and near zero at high frequencies.

Content slide 7: Analogies between systems.

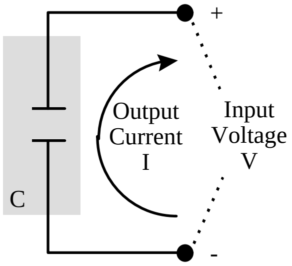
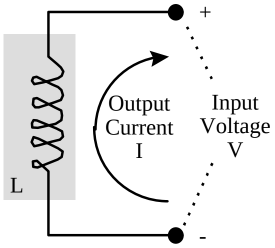
Analogies	**Potential Energy Storage**	**Energy Dissipation**	**Kinetic Energy Storage**
Electrical	I/V=Cs	I/V=1/R	I/V=1/(Ls)
Effort = Voltage drop, ΔV	Electrical Capacitance	Electrical Resistance	Inductance
Flow = Electric current, I			
Mechanical	V/F=Cs	V/F=1/B	V/F=1/(Ms)
Effort = Net Force, F	Mech. Compliance, C	Viscous Friction	Inertia
Flow = Velocity, V	1/C=K, spring constant	B = Friction	M = Mass
Fluid	q/ΔP=Cs	q/ΔP=1/R	q/ΔP =1/(Ms)
Effort = Pressure drop, ΔP	Fluid Compliance, C	Fluid Resistance, R	Fluid Inertia, M
Flow = fluid flow rate, q	1/C=K, fluid spring,	friction losses	(lumped)
(volumetric flow rate)	as in an accumulator	losses in an orifice	
Thermal	Q/ΔT=Cs	Q/ΔT =1/R	no known analogy
Effort = Temp. drop, ΔT	Thermal Heat Capacitance	Thermal Resistance	for thermal inertia
Flow = Heat flow rate, Q			

There are analogies to the electrical system building blocks in mechanical, fluid and thermal systems. The table above summaries the listed analogies. These analogies are classified as storing either potential or kinetic energy, or as dissipating energy. You can find these analogies by looking for the "*effort*" and "*flow*" variables. In electrical system, a voltage potential provides the "effort" and the flow is the flow of current. In a mechanical system, the effort is the net force and the flow is the velocity. In fluid systems the effort is the pressure potential and the flow is the volumetric flow rate. For thermal systems, the effort is the temperature difference between two points and the flow is the heat flow rate. Note that there is no analogy for the thermal inertia.

These similarities allow you to understanding a family of systems all having the same transfer function. For example, an electrical low pass filter comprised of a resistor and capacitor can be described by the same first order transfer function defined by a time constant and gain as the thermal transfer function that describes the temperature response of a thermocouple. Once you understand one system, learning other related systems is easier because of these analogies.

The analogies can also help in the construction of models comprised of mixed systems. For example, we can build a single transfer function model of an electromechanical servo system comprised of both electrical and mechanical components.

Note that while fluid and thermal systems tend to be nonlinear we can still use linear systems to model the dynamics if we limit our discussion to small perturbations around a nominal operating point.

Questions (1-2)	
The following questions will test your knowledge on the topics covered in this section.	
1) For the system shown, the current will lead, lag or be in phase with a sinusoidal input voltage?  	The voltage across a capacitor is proportional to the integral of the current that enters the capacitor. If we consider the voltage as the input and the current as the output, then the current is proportional to the time derivative of the voltage. The current leads the voltage by a phase angle of 90 degrees in a pure capacitor. $I(t) = C\frac{dV(t)}{dt}$ $\frac{\mathrm{I(s)}}{\mathrm{V(s)}} = \mathrm{Cs}$
2) For the system shown, the current will lead, lag or be in phase with a sinusoidal input voltage? Inductance  	The current in an inductor is proportional to the integral of the input voltage. Therefore, the current lags the voltage by a phase angle of 90 degrees in a pure inductor. $I(t) = \frac{1}{L}\int V(t)dt + I(0)$ $\frac{I(s)}{V(s)} = G_L(s) = \frac{1}{Ls}$